CAPM, Factor Models and APT

Corporate Finance and Incentives

Lars Jul Overby

Department of Economics
University of Copenhagen

October 2010

Capital asset pricing model

Assumptions

- Like for the Mean-Variance assumptions:
  - Markets are frictionless
  - Investors care only about their expected mean and variance of their returns over a given period
- Additional assumption required for CAPM:
  - Investors have homogeneous beliefs

Capital asset pricing model

Implications

The tangency portfolio is the same portfolio for all investors i.e. all investors hold the risky assets in the same relative proportions.

The tangency portfolio must be the market portfolio

The CAPM

$$\begin{aligned} \bar{r}_i &= r_f + \beta_i \left(\overline{R}_M - r_f \right) \\ \beta_i &= \frac{cov\left(\widetilde{r}_i, \widetilde{R}_M \right)}{var\left(\widetilde{R}_M \right)} \end{aligned}$$

Empirical results of the CAPM

Several problems of which the most noticeable are:

- Small Firms $\rightarrow$ Higher Return
  - Reduced since publication of the effect in early '80s
- Low Market Value/Book Value $\rightarrow$ Higher Return
  - Also reduced since publication in late '80s
- Momentum - past winning stocks outperform past losing stocks
  - Tendency still exists despite publication in early '90s
- Insignificance of Betas when above mentioned effects are accounted for

Factor models

The CAPM stems from a theoretical background - an equilibrium model
But it doesn't fit empirical asset returns well
What if assets are in fact exposed to other systematic risk factors, which affect the expected return hereon

- Factor models

Let's take a more statistical approach. Let's look at the actual behaviour of stock returns and their comovements.

Arbitrage Pricing Theory (APT)

- APT was conceived by Ross (1976)
- The model starts from a statistical point of view, not a theoretical one like the CAPM
- Idea: Not all types of risk are captured by the one market risk term of the CAPM
- There is a big common component to stock returns - the comovement with the market
- Beyond the market, some groups of stocks move together - like computer stocks, small stocks, utility stocks etc.
- Finally, the individual stocks have some idiosyncratic movement
- The claim of the APT is not that CAPM is incorrect – if CAPM's assumptions are correct it will hold
- But unlike CAPM, APT does not require that all investors only care about mean and variance
- Furthermore, if there is more than one source of systematic risk, maybe a richer model could give more insights

Factor models

A linear relationship between factors and assets is assumed
We have N assets and K factors, with $N > K$
The return of asset i is:

$$\widetilde{r}_i = \alpha_i + \beta_{i1}\widetilde{F}_1 + \beta_{i2}\widetilde{F}_2 + .. + \beta_{ik}\widetilde{F}_k + \widetilde{\varepsilon}_i$$

Where α_i is the intercept for the factor model, β_{ij} is asset $i's$ factor beta (factor sensitivity) to factor j, $\widetilde{F}_j$ is the level of factor j, and $\widetilde{\varepsilon}_i$ is an idiosyncratic risk adherent to asset i

Assumptions

In order for $\widetilde{\varepsilon}_i$ to be idiosyncratic (firm specific) it must hold that

$$\begin{aligned} E\left(\widetilde{\varepsilon}_i\right) &= 0 \\ cov\left(\widetilde{\varepsilon}_i, \widetilde{\varepsilon}_j\right) &= 0 \text{ for } i \neq j \\ cov\left(\widetilde{\varepsilon}_i, \widetilde{F}_h\right) &= 0 \end{aligned}$$

So the idiosyncratic risks are independent, meaning that they are diversifiable, which is key to the theory

Factor assumptions

For simplicity we assume that the factors in our model have been demeaned

$$E\left(F_h\right) = 0$$

This is in line with the idea that the factors proxy for new information about relevant variables

Often, we also work with uncorrelated factors

$$cov\left(\widetilde{F}_h, \widetilde{F}_k\right) = 0 \text{ for } h \neq k$$

Expected return

The question is, what is the expected return of asset i?

$$\begin{aligned} E\left(\widetilde{r}_i\right) &= E\left(\alpha_i + \beta_{i1}\widetilde{F}_1 + \beta_{i2}\widetilde{F}_2 + .. + \beta_{ik}\widetilde{F}_k + \widetilde{\varepsilon}_i\right) \\ &= \alpha_i \end{aligned}$$

But what is α_i?
The idea is, that the value of α_i depends on the exposure of asset i to the various factors. But how?
Two things matter

- How exposed is the asset to the risk factors?
    - Could fx be determined by running regressions of asset returns on the factors
- What does a unit of risk "cost"? The risk premium?

Pure factor portfolios

The expected return on asset i

$$E\left(\widetilde{r}_i\right) = r_f + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 + .. + \beta_{ik}\lambda_k = \alpha_i$$

We want to find the risk premiums, the $\lambda's$.
The easiest way to solve this is to create pure factor portfolios. In this way we can isolate the amount of risk in a given portfolio and link it to one single risk factor. This will allow us to decide on the risk premium on that particular risk factor.
Remember, when the k factors are uncorrelated

$$var\left(\widetilde{r}_i\right) = \beta_{i1}^2 var\left(\widetilde{F}_1\right) + \beta_{i2}^2 var\left(\widetilde{F}_2\right) + ... + \beta_{i3}^2 var\left(\widetilde{F}_3\right) + var\left(\widetilde{\varepsilon}_i\right)$$

Portfolio math

Use the fact that the factor beta of a portfolio on a given factor is the portfolio-weighted average of the individual securities' betas on that factor

$$
\begin{aligned}
\widetilde{R}_p &= \alpha_p + \beta_{p1}\widetilde{F}_1 + \beta_{p2}\widetilde{F}_2 + .. + \beta_{pk}\widetilde{F}_k + \widetilde{\varepsilon}_p \\
\alpha_p &= x_1\alpha_1 + x_2\alpha_2 + ... + x_N\alpha_N \\
\beta_{p1} &= x_1\beta_{11} + x_2\beta_{21} + ... + x_N\beta_{N1} \\
\beta_{p2} &= x_1\beta_{12} + x_2\beta_{22} + ... + x_N\beta_{N2} \\
& . \\
& .
\end{aligned}
$$

Assume the firm specific components can be diversified away (see result 6.1)

$$
\begin{aligned}
E\left(\widetilde{\varepsilon}_p\right) &= 0 \\
var\left(\widetilde{\varepsilon}_p\right) &\approx 0
\end{aligned}
$$

Pure factor portfolios

Create pure factor portfolios

$$\begin{aligned}
\widetilde{R}_{p1} &= \alpha_{p1} + 1 * \widetilde{F}_1 + 0 * \widetilde{F}_2 + .. + 0 * \widetilde{F}_k \\
\beta_{p11} &= x_{11}\beta_{11} + x_{12}\beta_{21} + ... + x_{1N}\beta_{N1} = 1 \\
\beta_{p12} &= x_{11}\beta_{12} + x_{12}\beta_{22} + ... + x_{1N}\beta_{N2} = 0 \\
& . \\
& . \\
& . \\
\beta_{p1k} &= x_{11}\beta_{1k} + x_{12}\beta_{2k} + ... + x_{1N}\beta_{Nk} = 0
\end{aligned}$$

Do this for each factor. This will give us portfolio weights for creating pure factor portfolios.

Pure factor portfolios

The expected return on a pure factor portfolio is then

$$\begin{aligned} E\left(\widetilde{R}_{p1}\right) &= r_f + 1 * \lambda_1 + 0 * \lambda_2 + .. + 0 * \lambda_k = \alpha_{p1} \\ &= x_{11}\alpha_1 + x_{12}\alpha_2 + ... + x_{1N}\alpha_N \end{aligned}$$

So the risk premium on F_1 is

$$\lambda_1 = x_{11}\alpha_1 + x_{12}\alpha_2 + ... + x_{1N}\alpha_N - r_f$$

Carry on to find value of all risk premiums

Tracking portfolio

Combine the pure factor portfolios and a risk free asset to construct tracking portfolios which have the same risk exposures as some given risky asset i.
The weights on the pure factor portfolios in the tracking portfolio are determined by the risk exposure of the risky asset i.
The portfolio weight on the risk free asset is such that the weights in the tracking portfolio sum to 1.
The expected return on the tracking portfolio is

$$E\left(\widetilde{R}_{TP}\right) = r_f + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 + .. + \beta_{ik}\lambda_k$$

Arbitrage

An arbitrage opportunity exists if the expected return on asset i differs from that of the tracking portfolio

$$\begin{aligned} E\left(\widetilde{r}_i\right) &= E\left(\alpha_i + \beta_{i1}\widetilde{F}_1 + \beta_{i2}\widetilde{F}_2 + .. + \beta_{ik}\widetilde{F}_k\right) = \alpha_i \\ &\neq r_f + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 + .. + \beta_{ik}\lambda_k = E\left(\widetilde{R}_{TP}\right) \end{aligned}$$

Arbitrage Pricing Theory

According to the arbitrage pricing theory, such arbitrage opportunities cannot exist
This implies that the risk premiums are the same for all assets

$$
\begin{aligned}
\lambda_{1i} &= \lambda_{1j} \text{ for } \forall\ i,j \\
\lambda_{2i} &= \lambda_{2j} \text{ for } \forall\ i,j \\
&. \\
&. \\
\lambda_{ki} &= \lambda_{kj} \text{ for } \forall\ i,j
\end{aligned}
$$

This gives us the APT model

$$\bar{r}_i = r_f + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 + .. + \beta_{ik}\lambda_k$$

for all investments with no firm-specific risk

Assumptions

- Returns can be described by a factor model
- There are no arbitrage opportunities
- There are a large number of securities, so it is possible to form portfolios that diversify the firm-specific risk of individual stocks
- The financial markets are frictionless

Implementation

APT can be implemented in three ways:

1. Using statistical methods to synthetically create “factors” that best fit the observed stock price variations
2. Using macroeconomic variables (after adjusting to make sure that the expected levels are 0)
3. Using firm-specific characteristics, such as firm size, as proxies for factor sensitivities

Factor analysis

- We will not go into how this is done, but. . .
- Covariances between stock returns are used to find the factor structure (factor levels and factor betas for each stock)
- Gives the best fit by construction
- However, makes interpretation impossible
- And in case something changes (like a company entering a foreign market, thus suddenly making it vulnerable to a certain FX rate), it is next to impossible to explain the implications for the factor sensitivities

Macroeconomic variables

Use macroeconomic variables as factors

- However, their levels must be adjusted:
- They should ideally have mean 0, meaning that, for instance, GDP growth can not be used directly as a factor
- The factor should be: GDP growth – consensus estimate of GDP growth
- This makes them hard to find – where to get the consensus estimates?
- On the other hand, this method has the benefit that interpretation is quite straightforward

Factors used

- Changes in monthly growth rate of GDP (reflects future demand for output)
- Changes in default risk premium, measured as spread between yields of AAA and Baa bonds (reflects concern about companies defaulting)
- Changes in the slope of the term structure (reflects expected future interest rates)
- Unexpected changes in the price level (alters the real value of contracts)
- Changes in expected inflation (reflects government policy and interest rates/discount rates)
- And many, many others...

Firm characteristics

Idea: certain firm characteristics are correlated with factor sensitivities (hard to measure) and therefore also to risk premia (easy to measure)

- Use these as proxies for factor sensitivities
- Transcends the problem of changing sensitivities and lack of intuition that factor analysis suffers from
- Also transcends the problem of factor changes having to be unexpected that using macroeconomics variables suffers from

Fama-French three factor model

One of the best models for explaining stock price returns - although it too has problems

Explanatory variables:

- Market return (CAPM beta)
- Market capitalization of the stock – small-cap stocks outperform large-cap stocks
- Market to book of the stock – low market to book stocks (value stocks) outperform high market to book stocks (growth stocks)
- Important: the interpretation is NOT that investors are compensated for holding small-cap stocks or low market to book stocks (then why hold anything else?), but rather that small-cap stocks are exposed to a certain risk that you are compensated for holding