Mean-Variance Optimization and CAPM
Corporate Finance and Incentives

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Practical Stuff

- Carsten and Benjamins homepage

Assumptions

- Investors care only about the means and variances of the returns of their portfolio over a given period. They prefer higher means and lower variances.
    - Investors are risk averse and risk can be described completely by return variance.
- Financial markets are frictionless.
    - Assets are tradeable at any price and in any quantity
    - There are no transaction costs
    - There are no taxes

If these conditions are satisfied, we can use the mean-standard deviation diagram to decide which portfolios are better than others.

Feasible set: The set of mean and standard deviation outcomes possible by combining the available assets.

Efficient portfolios: The portfolios for which the highest mean return is obtained for a given standard deviation.
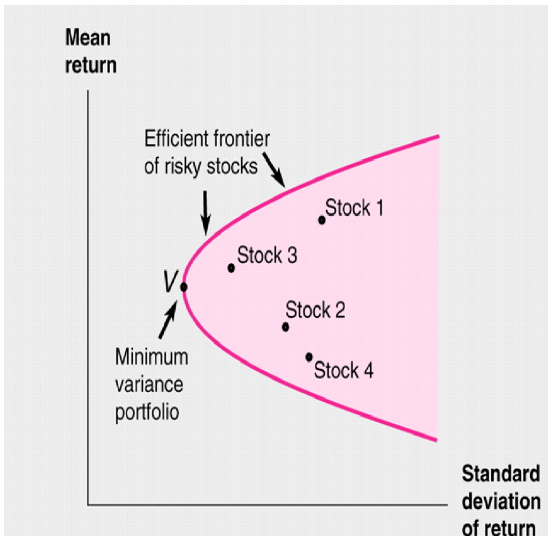
How do we find efficient portfolios?

"Choose a portfolio to minimize variance for a given mean return (μ)"

- Objective function: Portfolio variance
- Restrictions: Portfolio weights sum to 1 and mean return$= \mu$
- Minimize the objective function w.r.t. the portfolio weights, given the restrictions.

How do we find efficient portfolios?

- Portfolios of two assets fill out a hyperbolic curve passing through the two assets themselves
- All portfolios on the mean-variance efficient frontier can thus be formed as a weighted average of any two portfolios on the efficient frontier
    - Two-fund separation
- If we can find two portfolios on the efficient frontier, we can combine these two portfolios to create any other portfolio on the efficient frontier

$$\text{Efficient frontier} = w_1 Portfolio_1 + w_2 Portfolio_2$$
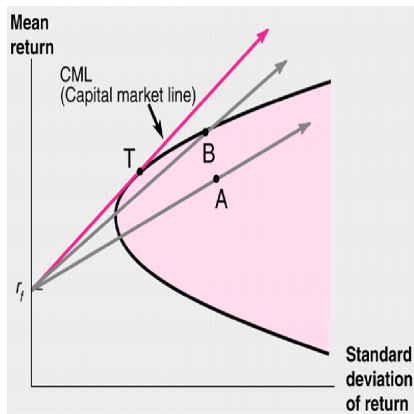
Introducing a risk-free asset

- So far we have looked at the efficient frontier with risky assets only.
- Introducing a risk-free asset expands the feasible set and changes the shape of the efficient frontier.
- A 100% holding of the risk-free asset must be the minimum variance portfolio, and must thus be on the efficient frontier.
- We only need to find one other portfolio on the efficient frontier to describe the entire frontier (two-fund separation theorem)

The tangency portfolio

- Combining a risk-free asset and a risky asset results in the mean-standard deviation relation being a straight line
- If we combine the risk-free asset with a portfolio from the feasible set of risky asset portfolios we thus get a straight line running through the two assets/portfolios.
- The straight line with the steepest slope must be the new efficient frontier.
- This is obtained by combining the risk-free asset and the tangency portfolio.

Capital Market Line

Introducing the risk-free asset creates the capital market line (CML). It is a weighted combination of the risk-free asset and the tangency portfolio

$$\overline{R}_p = r_f + \frac{\overline{R}_T - r_f}{\sigma_T}\sigma_p$$

Finding the tangency portfolio

The procedure for finding the tangency portfolio is similar to that used to identify the minimum variance portfolio.

The ratio of the risk premium of every stock and portfolio to its covariance with the tangency portfolio is constant

$$\frac{\overline{r}_i - r_f}{cov\left(\widetilde{r}_i, \widetilde{R}_T\right)} = k \text{ for all } i$$

1) Construct N equations consisting of the covariance of the N stocks' return with the tangency-portfolio (containing N unknown weights)
2) Rescale the weights so that they sum to 1

The security market line

The relation on the previous slide must also hold for the tangency portfolio itself

$$\frac{\overline{r}_i - r_f}{cov\left(\widetilde{r}_i, \widetilde{R}_T\right)} = \frac{\overline{R}_T - r_f}{cov\left(\widetilde{R}_T, \widetilde{R}_T\right)} = \frac{\overline{R}_T - r_f}{var\left(\widetilde{R}_T\right)}$$

This relation can be rewritten as

$$\begin{aligned} \overline{r}_i - r_f &= \frac{cov\left(\widetilde{r}_i, \widetilde{R}_T\right)}{var\left(\widetilde{R}_T\right)}\left(\overline{R}_T - r_f\right) \\ \overline{r}_i &= r_f + \beta_i\left(\overline{R}_T - r_f\right) \end{aligned}$$

where $\beta_i = \frac{cov\left(\widetilde{r}_i, \widetilde{R}_T\right)}{var\left(\widetilde{R}_T\right)}$

This is known as the security market line

Capital asset pricing model
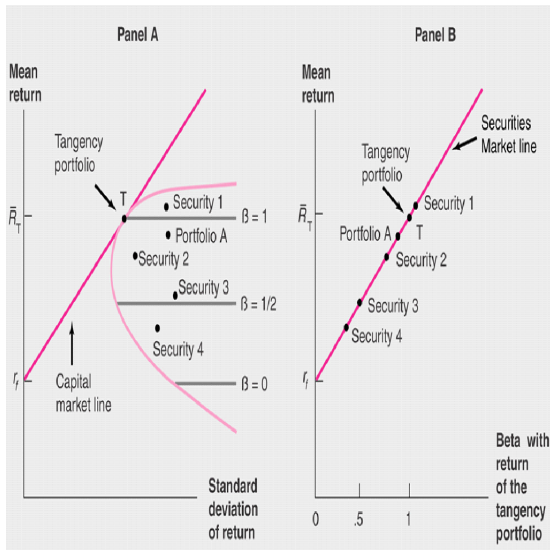
Assumptions

- Like for the Mean-Variance assumptions:
  - Markets are frictionless
  - Investors care only about their expected mean and variance of their returns over a given period
- Additional assumption required for CAPM:
  - Investors have homogeneous beliefs

Capital asset pricing model

Implications
The tangency portfolio is the same portfolio for all investors i.e. all
investors hold the risky assets in the same relative proportions.
The tangency portfolio must be the market portfolio

The CAPM

$$\bar{r}_i = r_f + \beta_i \left(\overline{R}_M - r_f \right)$$

$$\beta_i = \frac{cov\left(\widetilde{r}_i, \widetilde{R}_M \right)}{var\left(\widetilde{R}_M \right)}$$