Mean-Variance Optimization

Corporate Finance and Incentives

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Practical stuff

- Slides from exercise classes
- External presenters?

Risk and Return

The key objective when choosing to invest in financial assets is

- to maximize the expected return of the investment for a given level of risk
- to minimize the amount of risk of the investment for a given expected return

In order to achieve these objectives we need to

- compute the expected return of an investment
- quantify the risk of the investment

Harry Markowitz

The basis of modern portfolio theory is Harry Markowitz' mean-variance optimization theory[1]

The theory assumes that individuals

- minimize the return variance of an investment for any given level of expected return
- i.e. risk is quantified as the return variance (dispersion of return outcomes) of the investment

To work with this theory, we first need to look at some mathematics of portfolios

[1] Markowitz, Harry, 1952; "Portfolio Selection", Journal of Finance 7, 77-99.

Asset i

- Return on asset i: $R_i = \frac{P_{i,1}+D_{i,1}-P_{i,0}}{P_{i,0}} \sim \widetilde{r}_i$
- Expected return of asset i: $E\left(\widetilde{r}_i\right) = \sum_{s=1}^{S} q_s * \widetilde{r}_{i,s} \sim \overline{r}_i$

where q_s is the probability of scenario s and $\widetilde{r}_{i,s}$ is the return on asset i is scenario s occurs.

- Expected value of a constant times a return: $E\left(x\widetilde{r}_i\right) = xE\left(\widetilde{r}_i\right)$
- Return variance of asset i:
  $var\left(\widetilde{r}_i\right) = \sum_{s=1}^{S} q_s * \left[\left(R_{i,s} - E\left(R_i\right)\right)^2\right] = E\left[\left(\widetilde{r}_i - \overline{r}_i\right)^2\right] = \sigma_i^2$
- Variance of constant times a return: $var\left(x\widetilde{r}_i\right) = x^2 var\left(\widetilde{r}_i\right)$
- Standard deviation: $\sigma\left(x\widetilde{r}_i\right) = x\sigma\left(\widetilde{r}_i\right)$

Expected return and variance of asset i

State	$\widetilde{r}_i$	q_s
A	0.01	0.25
B	0.04	0.50
C	0.08	0.25

Portfolio returns

- Portfolio weights: $x_j = \frac{\text{value of holding of asset j}}{\text{total value of portfolio}}$
- Portfolio returns: $\widetilde{R}_p = x_1\widetilde{r}_1 + x_2\widetilde{r}_2 + ... + x_N\widetilde{r}_N = \sum_{i=1}^{N} x_i\widetilde{r}_i$
- Expected value of the sum or difference of two returns: $E\left(\widetilde{r}_i + \widetilde{r}_j\right) = E\left(\widetilde{r}_i\right) + E\left(\widetilde{r}_j\right)$ and $E\left(\widetilde{r}_i - \widetilde{r}_j\right) = E\left(\widetilde{r}_i\right) - E\left(\widetilde{r}_j\right)$
- Expected return of 2 asset portfolio: $E\left(\widetilde{R}_p\right) = E\left(x_i\widetilde{r}_i + x_j\widetilde{r}_j\right) = x_iE\left(\widetilde{r}_i\right) + x_jE\left(\widetilde{r}_j\right)$
- Expected return of N asset portfolio: $E\left(\widetilde{R}_p\right) = \sum_{i=1}^{N} x_i\bar{r}_i$

Portfolio returns

State	q_s	$\widetilde{r}_1$	$\widetilde{r}_2$	$\widetilde{r}_3$
A	0.25	0.01	0.10	0.01
B	0.50	0.04	0.03	0.06
C	0.25	0.08	0.07	0.09
$E(\widetilde{r}_i)$		0.0425	0.0575	0.055
$var(\widetilde{r}_i)$		0.000619	0.000869	0.000825
$std.dev.$		0.0249	0.0295	0.0287
x_i		0.30	0.45	0.25

Two ways to compute expected portfolio return

- Compute portfolio returns in each state and take expectation
- Compute weighted average of expected asset returns

Covariances and correlations

- Covariance of two returns: $\sigma_{ij} = E\left[\left(\widetilde{r}_i - \overline{r}_i\right)\left(\widetilde{r}_j - \overline{r}_j\right)\right] = cov(\widetilde{r}_i, \widetilde{r}_j)$
- Correlation between two returns: $\rho\left(\widetilde{r}_i, \widetilde{r}_j\right) = \frac{cov(\widetilde{r}_i, \widetilde{r}_j)}{\sigma_i \sigma_j} = \rho_{ij}$

If $\rho_{ij} = 1$, return i and j are perfectly correlated.
If $\rho_{ij} = -1$, return i and j are perfectly negatively correlated.

Variance of a 2-asset portfolio

$$
\begin{aligned}
var(x_i\tilde{r}_i + x_j\tilde{r}_j) &= E\left\{\left[x_i\tilde{r}_i + x_j\tilde{r}_j - (x_i\bar{r}_i + x_j\bar{r}_j)\right]^2\right\} \\
&= E\left\{\left[x_i\left(\tilde{r}_i - \bar{r}_i\right) + x_j\left(\tilde{r}_j - \bar{r}_j\right)\right]^2\right\} \\
&= E\left[x_i^2\left(\tilde{r}_i - \bar{r}_i\right)^2 + x_j^2\left(\tilde{r}_j - \bar{r}_j\right)^2 + 2x_ix_j\left(\tilde{r}_i - \bar{r}_i\right)\left(\tilde{r}_j - \bar{r}_j\right)\right. \\
&= x_i^2E\left[\left(\tilde{r}_i - \bar{r}_i\right)^2\right] + x_j^2E\left[\left(\tilde{r}_j - \bar{r}_j\right)^2\right] \\
&\quad +2x_ix_jE\left[\left(\tilde{r}_i - \bar{r}_i\right)\left(\tilde{r}_j - \bar{r}_j\right)\right] \\
&= x_i^2 var(\tilde{r}_i) + x_j^2 var\left(\tilde{r}_j\right) + 2x_ix_j cov(\tilde{r}_i, \tilde{r}_j) \\
&= x_i^2\sigma_i^2 + x_j^2\sigma_j^2 + 2x_ix_j\sigma_{ij} \\
&= x_i^2\sigma_i^2 + x_j^2\sigma_j^2 + 2x_ix_j\rho_{ij}\sigma_i\sigma_j
\end{aligned}
$$

Given positive portfolio weights on two assets, the lower the correlation between returns, the lower the variance of the portfolio.

Variance-covariance-correlation matrix[2]

Asset	1	2	3
1	0.000619	−0.00019	0.000688
2	−0.264	0.000869	−0.00044
3	0.962	−0.517	0.000825

[2] Variance on diagonal, covariances above diagonal, correlations below diagonal.

Variance of an N-asset portfolio

$$
\begin{aligned}
var\left(\sum_{j=1}^{N} x_i \widetilde{r}_i\right) &= \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \rho_{ij} \sigma_i \sigma_j
\end{aligned}
$$

Markowitz' mean-variance optimization

- We now have the tools to compute portfolio returns and variances.
- Markowitz said, that the only thing investors care about is the relation between these two.
- Investors wish to achieve the highest possible expected return for a given level of return variance.

Two asset portfolio

$$\begin{aligned} E\left(\widetilde{R}_p\right) &= x_i E\left(\widetilde{r}_i\right) + x_j E\left(\widetilde{r}_j\right) \\ \sigma_p^2 &= x_i^2 \sigma_i^2 + x_j^2 \sigma_j^2 + 2 x_i x_j \rho_{ij} \sigma_i \sigma_j \end{aligned}$$

Assume we have a risk-free asset and a risky asset.

$$\begin{aligned} E\left(\widetilde{R}_p\right) &= x_f r_f + x_j E\left(\widetilde{r}_j\right) \\ \sigma_p^2 &= x_f^2 * 0 + x_j^2 \sigma_j^2 + 2 x_f x_j * 0 * 0 * \sigma_j = x_j^2 \sigma_j^2 \\ \sigma_p &= x_j \sigma_j \quad \text{if } x_j \geq 0 \\ \sigma_p &= -x_j \sigma_j \quad \text{if } x_j < 0 \\ E\left(\widetilde{R}_p\right) &= r_f + \frac{E\left(\widetilde{r}_j\right) - r_f}{\sigma_j} \sigma_p \text{ if } x_j \geq 0 \\ E\left(\widetilde{R}_p\right) &= r_f - \frac{E\left(\widetilde{r}_j\right) - r_f}{\sigma_j} \sigma_p \text{ if } x_j < 0 \end{aligned}$$

Two risky assets

Two perfectly negatively correlated assets: $\rho = -1$

$$
\begin{aligned}
E\left(\widetilde{R}_p\right) &= x_1 E\left(\widetilde{r}_1\right) + x_2 E\left(\widetilde{r}_2\right) \\
\sigma_p^2 &= x_1^2 * \sigma_1^2 + x_2^2 \sigma_2^2 + 2 x_1 x_2 * (-1) * \sigma_1 * \sigma_2 \\
&= \left(x_1 \sigma_1 - x_2 \sigma_2\right)^2 \\
\sigma_p &= x_1 \sigma_1 - x_2 \sigma_2 \text{ if } x_1 \sigma_1 - x_2 \sigma_2 \geq 0 \\
\sigma_p &= -x_1 \sigma_1 + x_2 \sigma_2 \text{ if } x_1 \sigma_1 - x_2 \sigma_2 < 0
\end{aligned}
$$

if $x_1\sigma_1 - x_2\sigma_2 \geq 0$

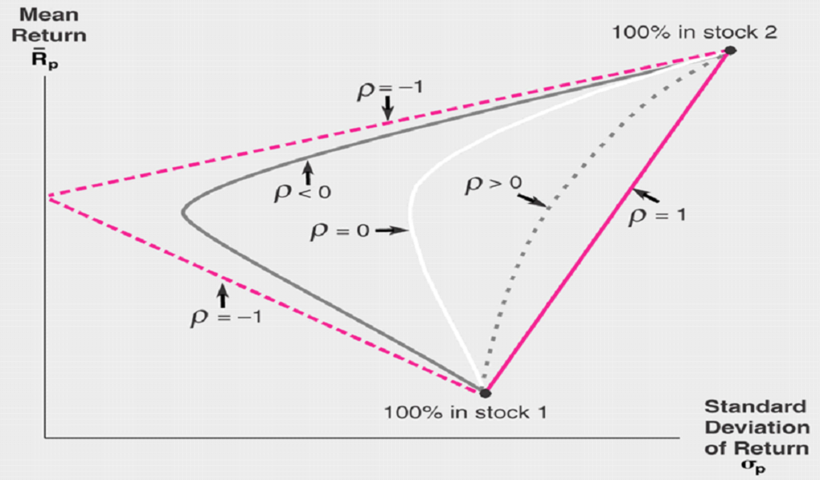
$$
\begin{aligned}
\sigma_p &= (1-x_2)\,\sigma_1 - x_2\sigma_2 = \sigma_1 - x_2\,(\sigma_2+\sigma_1) \Leftrightarrow \\
x_2 &= \frac{-\sigma_p}{\sigma_2+\sigma_1} + \frac{\sigma_1}{\sigma_2+\sigma_1} \\
E\left(\widetilde{R}_p\right) &= \left(1 + \frac{\sigma_p}{\sigma_2+\sigma_1} - \frac{\sigma_1}{\sigma_2+\sigma_1}\right) E\left(\widetilde{r}_1\right) \\
&\quad + \left(\frac{-\sigma_p}{\sigma_2+\sigma_1} + \frac{\sigma_1}{\sigma_2+\sigma_1}\right) E\left(\widetilde{r}_2\right) \\
&= E\left(\widetilde{r}_1\right) + \left(E\left(\widetilde{r}_2\right) - E\left(\widetilde{r}_1\right)\right)\frac{\sigma_1}{\sigma_2+\sigma_1} - \frac{E\left(\widetilde{r}_2\right) - E\left(\widetilde{r}_1\right)}{\sigma_2+\sigma_1}\sigma_p
\end{aligned}
$$

Two risky assets

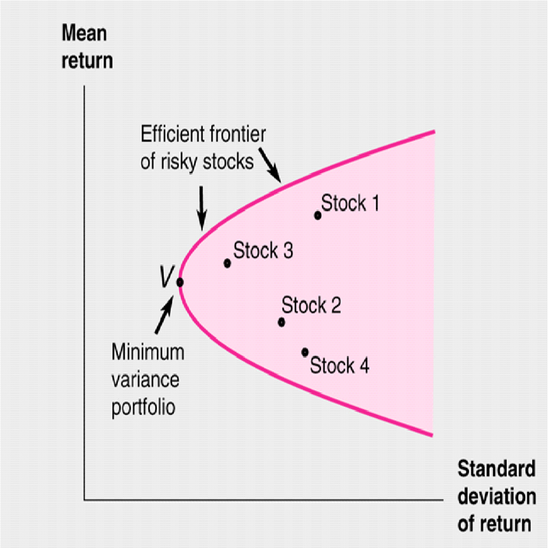
if $x_1\sigma_1 - x_2\sigma_2 < 0$

$$\begin{aligned}
\sigma_p &= (x_2 - 1)\,\sigma_1 + x_2\sigma_2 = -\sigma_1 + x_2\,(\sigma_2 + \sigma_1) \Leftrightarrow \\
x_2 &= \frac{\sigma_p}{\sigma_2 + \sigma_1} + \frac{\sigma_1}{\sigma_2 + \sigma_1} \\
E\left(\widetilde{R}_p\right) &= \left(1 - \frac{\sigma_p}{\sigma_2 + \sigma_1} - \frac{\sigma_1}{\sigma_2 + \sigma_1}\right) E\left(\widetilde{r}_1\right) \\
&\quad + \left(\frac{\sigma_p}{\sigma_2 + \sigma_1} + \frac{\sigma_1}{\sigma_2 + \sigma_1}\right) E\left(\widetilde{r}_2\right) \\
&= E\left(\widetilde{r}_1\right) + \left(E\left(\widetilde{r}_2\right) - E\left(\widetilde{r}_1\right)\right) \frac{\sigma_1}{\sigma_2 + \sigma_1} + \frac{E\left(\widetilde{r}_2\right) - E\left(\widetilde{r}_1\right)}{\sigma_2 + \sigma_1}\sigma_p
\end{aligned}$$

Two risky assets



Four risky assets



Minimum variance portfolio (without a risk-free asset)

- The portfolio of a group of stocks that minimizes return variance is the portfolio with a return that has an equal covariance with every stock return[3].

[3] See proof in section 4.8.

Finding the Minimum Variance Portfolio with N stocks

1) Construct N equations consisting of the covariance of the N stocks' return with the MV-Portfolio (containing N unknown weights)

$$
\begin{aligned}
cov\left(\widetilde{r}_1, \widetilde{R}_p\right) &= cov\left(\widetilde{r}_1, \sum_{i=1}^{N} w_i \widetilde{r}_i\right) = k \\
cov\left(\widetilde{r}_2, \sum_{i=1}^{N} w_i \widetilde{r}_i\right) &= k \\
&\dots \\
&\dots \\
cov\left(\widetilde{r}_N, \sum_{i=1}^{N} w_i \widetilde{r}_i\right) &= k
\end{aligned}
$$

2) Rescale the weights so that they sum to 1 $x_i = \frac{w_i}{\sum w_i}$

See examples in section 4.9