Derivatives III

Corporate Finance and Incentives

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The Black-Scholes Model (1973)

Assume we take the binomial model to the limit, and divide the options life into an infinite number of subperiods.
This gives us continuously adjusting stock prices which we assume to be lognormally distributed.
The principle of option evaluation is still the same as in the simple binomial models, but we now have to rebalance the replicating portfolio continuously.

The Black-Scholes Model

Most critical assumptions:

- Underlying asset price follows a continuous time diffusion (no jumps) - geometric Brownian motion
- Volatility is constant and known
- The price of the underlying at expiration is lognormally distributed $\rightarrow$ returns are normally distributed
- Markets are frictionless

The Black-Scholes model provides a relatively simple way to handle this continuous rebalancing and value options in such a setup.
Value of an option:

$$
\begin{aligned}
call &= SN(d_1) - Ke^{-r_f t}N(d_2) \\
put &= Ke^{-r_f t}N(-d_2) - SN(-d_1) \\
d_1 &= \frac{\ln(S/K) + \left(r_f + \sigma^2/2\right)t}{\sigma\sqrt{t}} \\
d_2 &= d_1 - \sigma\sqrt{t} \\
N(d) &= \text{cumulative normal probability density function} \\
S &= \text{stock price now} \\
K &= \text{exercise price of option} \\
t &= \text{ number of periods to exercise date} \\
\sigma &= \text{standard deviation (volatility) per period of stock return} \\
r_f &= \text{risk-free interest rate per period}
\end{aligned}
$$

Estimating volatilities

The one parameter in the Black-Scholes model which cannot be directly observed is the volatility of the price of the underlying asset.

- The standard deviation of the return on the underlying over one period (typically 1 year), when the return is expressed using continuous compounding - σ

Two ways to estimate volatility

- Using historical data
  - find an estimate of the historical volatility of the return of the underlying asset
- Implied volatility
  - use market option prices on actively traded options to back out implied volatilities

European puts and calls - dividend paying stock

The option holder is not entitled to dividends paid on the stock during the option period.
When dividends are paid, the stock price falls proportionally to the size of the dividends.
When using the Black-Scholes model to value European options on dividend paying stocks, we must reduce the initial stock price by the present value of the dividends expected to be paid out during the life of the option - $S_0 e^{-div*T}$

Options on stocks paying a continuous dividend yield

$$\begin{aligned}
call &= Se^{-div*t}N(d_1) - Ke^{-r_f t}N(d_2) \\
put &= Ke^{-r_f t}N(-d_2) - Se^{-div*t}N(-d_1) \\
d_1 &= \frac{\ln\left(S/K\right) + \left(r_f - div + \sigma^2/2\right)t}{\sigma\sqrt{t}} \\
d_2 &= d_1 - \sigma\sqrt{t} \\
N(d) &= \text{cumulative normal probability density function} \\
S &= \text{stock price now} \\
K &= \text{exercise price of option} \\
t &= \text{number of periods to exercise date} \\
\sigma &= \text{standard deviation (volatility) per period of stock return} \\
div &= \text{continuous dividend yield} \\
r_f &= \text{risk-free interest rate per period}
\end{aligned}$$

Currency options

- Companies may be interested in managing foreign currency risk by entering into currency options, instead of forward contracts.
- As with stock options, we assume that the spot exchange rate of a currency pair follows a geometric brownian motion.
- Define the spot exchange rate as the value of one unit of the foreign currency measured in the domestic currency.
- A foreign currency is analogous to a stock providing a known dividend yield - the owner of the foreign currency receives a "dividend yield" equal to the risk-free interest rate in the foreign currency - r_{fo}

Currency options

$$
\begin{aligned}
call &= Se^{-r_{fo} t} N(d_1) - Ke^{-r_d t} N(d_2) \\
put &= Ke^{-r_d t} N(-d_2) - Se^{-r_{fo} * t} N(-d_1) \\
d_1 &= \frac{\ln(S/K) + \left(r_d - r_{fo} + \sigma^2/2\right) t}{\sigma\sqrt{t}} \\
d_2 &= d_1 - \sigma\sqrt{t} \\
N(d) &= \text{cumulative normal probability density function} \\
S &= \text{spot exchange rate now} \\
K &= \text{exercise price of option} \\
t &= \text{ number of periods to exercise date} \\
\sigma &= \text{standard deviation (volatility) per period of stock return} \\
div &= \text{continuous dividend yield} \\
r_{fo} &= \text{foreign risk-free interest rate per period} \\
r_d &= \text{domestic risk-free interest rate per period}
\end{aligned}
$$

Options on forwards - Black's model

- Fischer Black extended the Black-Scholes model to value options on forward contracts
- The key insight is that forward prices behave in the same way as a stock paying a continuous dividend yield given by the domestic risk-free interest rate r_f - $S_0 = F_0 e^{-r_f T}$

Options on forwards - Black's model

$$
\begin{aligned}
call &= Fe^{-r_f * t} N(d_1) - Ke^{-r_f t} N(d_2) \\
put &= Ke^{-r_f t} N(-d_2) - Fe^{-r_f * t} N(-d_1) \\
d_1 &= \frac{\ln(F/K) + \sigma^2 * t/2}{\sigma\sqrt{t}} \\
d_2 &= d_1 - \sigma\sqrt{t} \\
N(d) &= \text{cumulative normal probability density function} \\
F &= \text{forward price now} \\
K &= \text{exercise price of option} \\
t &= \text{ number of periods to exercise date} \\
\sigma &= \text{standard deviation (volatility) per period of stock return} \\
r_f &= \text{risk-free interest rate per period}
\end{aligned}
$$

The Greeks

The sensitivities of the option premium w.r.t the underlying factors behind option pricing are commonly referred to as the greeks:

- The Delta
- The Gamma
- The Vega
- The Theta
- The Rho(s)

These are also useful for financial institutions that sell options, as risk management tools.

The Delta

Definition: The sensitivity of the premium of an option w.r.t. to the underlying asset

Describes the option's exposure to the underlying asset and thus the required amount of shares needed to hedge the risk - Delta neutral

The Delta is always positive for Calls and negative for Puts.

The Gamma

Definition: The sensitivity of the Delta w.r.t. to the underlying asset (i.e. the convexity of the premium w.r.t. the underlying asset)

Describes how the Delta changes when the underlying asset changes

The Gamma is always positive for vanilla options (Puts and Calls)

Describes what dynamic Delta Hedging is required to remain Delta neutral - how often we must rebalance

Definition: The sensitivity of the premium w.r.t. to the volatility of the underlying asset

The Vega is always positive for vanilla options (Puts and Calls)

Hence, the higher the volatility the higher the premium. This is due to the payout function of an option with limited downside and unlimited upside

Definition: The sensitivity of the premium w.r.t. to the passing of time

This reflects the loss in time value - the change in the option premium when the time to maturity decreases.

Theta is usually negative for an option

- The longer time to maturity, the more variability is possible in the value of the underlying.

In some cases it may be positive fx some in-the-money put options on non-dividend paying stocks

- The longer time to maturity makes the present value of the future cash **inflow** from the put smaller

And some European call options on dividend paying stocks

- We can't exercise early and avoid the drop in stock prices that occurs when a dividend is paid - hence a short dated call option may be more valuable than a long dated call option.

The Rho(s)

Definition: The sensitivity of the premium w.r.t. the interest rate

This is positive for call options and negative for put options -

- Call option - the present value of the future cash **outflow** of K on an in-the-money option becomes smaller
- Put option - the present value of the future cash **inflow** of K on an in-the-money option become smaller

But remember, this holds only when all other factors are held fixed. If the interest rate increases, the value of the underlying stock is likely to decrease - so the net effect may differ from that given above.
For currency options there are two Rho's - one with respect to the domestic interest rate and one with respect to the foreign interest rate.

The effect on the option price of increasing one variable

	Call Option	Put Option
Price of underlying $(S) \uparrow$	$\uparrow$	$\downarrow$
Exercise price $(K) \uparrow$	$\downarrow$	$\uparrow$
Interest rate $(r_f) \uparrow$	$\uparrow$	$\downarrow$
Time to expiration $(T) \uparrow$	$?\,(\uparrow)$	$?\,(\uparrow)$
Volatility of underlying $(\sigma) \uparrow$	$\uparrow$	$\uparrow$
Dividends	$\downarrow$	$\uparrow$

(American option value in parentheses)

Volatility smiles

If we back out the implied volatilities of stock options priced in the market, we often observe a smile - instead of a straight line.