Derivatives II: More about options

Corporate Finance and Incentives

Lars Jul Overby

Department of Economics
University of Copenhagen

October 2010

Generalization

Let the initial stock price be S_0 and the current price of an option on the stock V. The expiration date of the option is T.
During the life of the option, the stock can either move up to a new price of $S_0 u$ or down to $S_0 d$ $(u > 1; d < 1)$.
If the stock moves up, let the payoff from the option be V_u.
If the stock moves down, let the payoff from the option be V_d.
The value of the portfolio (long Δ stocks and short 1 option) at expiration will be

$$S_0 u\Delta - V_u$$

if the stock moves up. And

$$S_0 d\Delta - V_d$$

if the stock moves down.

These will be equal when

$$\Delta = \frac{V_u - V_d}{S_0 u - S_0 d}$$

When delta takes on this value, the portfolio is riskless and must earn the risk-free interest rate. At time 0 the value of the portfolio is therefore

$$\left(S_0 u\Delta - V_u\right)\left(1 + r_f\right)^{-T}$$

The initial cost of the portfolio is

$$S_0\Delta - V$$

Hence

$$
\begin{aligned}
(S_0 u\Delta - V_u)(1+r_f)^{-T} &= S_0\Delta - V \\
V &= S_0\Delta - (S_0 u\Delta - V_u)(1+r_f)^{-T} \\
&= S_0 \frac{V_u - V_d}{S_0 u - S_0 d} \\
&\quad - \left(S_0 u \frac{V_u - V_d}{S_0 u - S_0 d} - V_u \right)(1+r_f)^{-T} \\
&= (1+r_f)^{-T}\left[qV_u + (1-q)V_d\right] \\
q &= \frac{(1+r_f)^{T} - d}{u-d}
\end{aligned}
$$

we can interpret q as the probability of an up movement in the stock and $1-q$ as the probability of a down movement.
The value of the option today is then the expected future value discounted at the risk-free rate.

Irrelevance of the stocks expected return

- The option pricing formula does not involve the probabilities of the stock price moving up or down.
- We are not valuing the option in absolute terms, but in terms of the price of the underlying stock.
- The probabilities of future up and down movements in the stock are already incorporated in the stock price
    - no need to account for this again

Risk-neutral valuation

The expected return on the stock, when q is assumed to be the probability of an up movement

$$\begin{aligned} E\left(S_T\right) &= qS_0u + (1-q)\,S_0d \\ &= qS_0(u-d) + S_0d \\ &= \frac{\left(1+r_f\right)^T - d}{u-d} S_0(u-d) + S_0d \\ &= S_0\left(1+r_f\right)^T \end{aligned}$$

Setting the probability of an up movement in the stock to q is thus equivalent to the rate of return on the stock being the risk-free rate. This thus assumes that individuals are indifferent to risk - a risk-neutral world.

Using the above q measure is an example of risk-neutral valuation.

Determining u and d

How do we determine the size of the possible up and down movements in the stock.

If It can be shown that these depend on the standard deviation or volatility of the stock return σ in the following manner

$$\begin{aligned} 1 + \text{upside change} &= u = e^{\sigma\sqrt{h}} \\ 1 + \text{downside change} &= d = \frac{1}{u} \end{aligned}$$

where h is the time interval, as a fraction of a year, over which the movement is observed (assuming σ is the annual volatility rate).

In our example $u = \frac{80}{60} = 1.3333$

Thus we have an annualized volatility of the stock of

$$\begin{aligned} 1.3333 &= e^{\sigma\sqrt{\frac{2}{3}}} \\ \sigma &= \frac{\ln(1.3333)}{\sqrt{\frac{2}{3}}} = 0.3523 \end{aligned}$$

Generalized binomial model

For the binomial model to be in any way useful, there must be more than one period, since a stock price can take on more than 2 values
However, adding more states with just a single time period is problematic, since it makes the market incomplete, meaning that finding unique derivatives prices is impossible

- we can't create replicating portfolios

Instead, we add more time periods to the model

- slice up the life of the option into smaller periods

Generalized binomial model

Instead of assuming two possible stock price outcomes in 8 months, assume there are two time steps over the 8 month period.
At each node at each time step, there are two possible stock price moves.
This does not change the valuation method, but requires rebalancing of the replicating portfolio along the way.

- Dynamically adjusted portfolios - the delta changes

The principle can be extended to a finer and finer division of the time period over which the option runs

- increasing the number of possible stock prices at expiration - until we reach the point where stock prices change continuously

When valuing the option we use backwards induction

Two-stage binomial model

Stock: $S_0 = \$60, \sigma = 35.23\%$
Put option: $K = \$60$, $T = 2/3$
$r_f = 1.5\% p.a.$
Value option by two-stage binomial model

$$
\begin{aligned}
1 + \text{upside change (4 months)} &= u_4 = e^{0.3523 * \sqrt{\frac{1}{3}}} = 1.2256 \\
1 + \text{downside change (4 months)} &= d_4 = \frac{1}{u_4} = 0.8159
\end{aligned}
$$

$p = \frac{(1+r_f)^h - d}{u-d} = \frac{(1+0.015)^{\frac{1}{3}} - 0.8159}{1.2256 - 0.8159} = 0.4615$

Stock price and option development

$$\begin{aligned}
S_0 &= \$60 \\
S_{4,u} &= \$60 * 1.2256 = \$73.54 \\
S_{4,d} &= \$60 * 0.8159 = \$48.95 \\
S_{8,u,u} &= \$60 * 1.2256 * 1.2256 = \$90.13 \\
S_{8,u,d} &= S_{8,d,u} = \$60 * 1.2256 * 0.8159 = \$60.00 \\
S_{8,d,d} &= \$60 * 0.8159 * 0.8159 = \$39.94
\end{aligned}$$

$$\begin{aligned}
p_{8,u,u} &= V_{u,u} = \$0 \\
p_{8,u,d} &= V_{u,d} = \$0 \\
p_{8,d,d} &= V_{d,d} = \$60 - \$39.94 = \$20.06
\end{aligned}$$

Option value at month 4

$$\text{Option delta} = \frac{0 - 20.06}{60 - 39.94} = -1$$

Construct a leveraged position in delta shares which gives an identical payoff to the option:

	$S_{8,d,d}$	$S_{8,d,u}$
Sell 1 share	$-\$39.94$	$-\$60.00$
Invest PV of \$60	\$60.00	\$60.00
Total payoff	\$20.06	\$0

Value of put in month 4:

$put_{4,d} = -\$48.95 + \$60 * (1 + 0.015)^{-\frac{1}{3}} = \$10.75$ or

$put_{4,d} = (1 + r_f)^{-h} [pV_{u,d} + (1 - p) V_{d,d}] =$

$(1 + 0.015)^{-\frac{1}{3}} [0.4615 * \$0 + (1 - 0.4615) * \$20.06] = \$10.75$

Option value now

$$\text{Option delta} = \frac{0 - 10.75}{73.54 - 48.95} = -0.437$$

Construct a leveraged position in delta shares which gives an identical payoff to the option:

	$S_{4,d}$	$S_{4,u}$
Sell 0.437 shares	$-\$21.39$	$-\$32.14$
Invest PV of \$32.14	\$32.14	\$32.14
Total payoff	\$10.75	\$0

Value of put now: $put = -\$60 * 0.437 + \$32.14 * (1 + 0.015)^{-\frac{1}{3}} = \$5.76$
or
$put = (1 + 0.015)^{-\frac{1}{3}} [0.4616 * \$0 + (1 - 0.4616) * \$10.75] = \$5.76$

Generalized binomial model

Two step

$$
\begin{aligned}
V_u &= (1+r_f)^{-h}\left[qV_{u,u} + (1-q)V_{u,d}\right] \\
V_d &= (1+r_f)^{-h}\left[qV_{u,d} + (1-q)V_{d,d}\right] \\
V &= (1+r_f)^{-h}\left[qV_u + (1-q)V_d\right] \\
V &= (1+r_f)^{-2h}\left[q^2 V_{u,u} + 2q(1-q)V_{u,d} + (1-q)^2 V_{dd}\right]
\end{aligned}
$$

Generalized to N-steps - see formula 8.2 in book $(\pi = q)$

The Black-Scholes Model (1973)

Assume we take the binomial model to the limit, and divide the options life into an infinite number of subperiods.
This gives us continuously adjusting stock prices which we assume to be lognormally distributed.
The principle of option evaluation is still the same as in the simple binomial models, but we now have to rebalance the replicating portfolio continuously.

The Black-Scholes Model

Most critical assumptions:

- Underlying asset price follows a continuous time diffusion (no jumps)
- Volatility is constant and known
- Returns are lognormally distributed
- Markets are frictionless

The Black-Scholes model provides a relatively simple way to handle this continuous rebalancing and value options in such a setup.
Value of an option:

$$
\begin{aligned}
call &= SN(d_1) - Ke^{-r_f t}N(d_2) \\
put &= Ke^{-r_f t}N(-d_2) - SN(-d_1) \\
d_1 &= \frac{\ln(S/K) + \left(r_f + \sigma^2/2\right)t}{\sigma\sqrt{t}} \\
d_2 &= d_1 - \sigma\sqrt{t} \\
N(d) &= \text{cumulative normal probability density function} \\
S &= \text{stock price now} \\
K &= \text{exercise price of option} \\
t &= \text{number of periods to exercise date} \\
\sigma &= \text{standard deviation (volatility) per period of stock return} \\
r_f &= \text{interest rate per period}
\end{aligned}
$$

Stock: $S_0 = \$60, \sigma = 35.23\%$
Put option: $K = \$60$, $T = 2/3$
$r_f = 1.5\% p.a.$ continuously compounded
Black-Scholes formula for a put

$$
\begin{aligned}
put &= Ke^{-r_f t} N(-d_2) - SN(-d_1) \\
d_1 &= \frac{\ln (S/K) + \left(r_f + \sigma^2/2\right) t}{\sigma \sqrt{t}} \\
&= \frac{\ln (60/60) + \left(0.015 + 0.3523^2/2\right) 2/3}{0.3523\sqrt{2/3}} \\
&= 0.1786 \\
d_2 &= d_1 - \sigma\sqrt{t} = 0.1786 - 0.3523\sqrt{2/3} = -0.1091 \\
N(-d_1) &= 0.4291 \\
N(-d_2) &= 0.5434 \\
put &= 60e^{-0.015*2/3} 0.5434 - 60 * 0.4291 = 6.534
\end{aligned}
$$

As the number of intervals used in the binomial model increases, the option price converges towards the option price from the Black-Scholes model.
The Black-Scholes formula is generally quicker and more accurate to use than the binomial model.
However, there are a number of cases in which the Black-Scholes model cannot be applied.
The Black-Scholes model cannot handle early exercise.

American puts and calls - no dividends

American calls should never be exercised early $\rightarrow$ we can use the Black-Scholes model

It can sometimes pay to exercise an American put option early, if it is sufficiently in the money $\rightarrow$ we cannot use the Black-Scholes model. Instead, we use the step-by-step binomial model and check at each step whether it is profitable to exercise.

European puts and calls - dividend paying stock

The option holder is not entitled to dividends paid on the stock during the option period.
When dividends are paid, the stock price falls proportionally to the size of the dividends.
When using the Black-Scholes model to value European options on dividend paying stocks, we must reduce the initial stock price by the present value of the dividends expected to be paid out during the life of the option.

American puts and calls - dividend paying stock

As was the case for American puts on non-dividend paying stocks, it may sometimes pay to exercise an American put on a dividend paying stock early $\rightarrow$ we cannot use the Black-Scholes model.

An American call option on a dividend paying stock should only be exercised early if the dividend you gain by exercising is bigger than the interest you loose by paying the exercise price early.
Use the binomial model to check if it is profitable to exercise just before the ex-dividend date.