Derivatives I

Corporate Finance and Incentives

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Derivatives

A financial contract written on an underlying asset
Can basically take any form and be written on any underlying asset

Contents of a derivative contract:

- Specifies the underlying asset
- Specifies payout function and timing of such
- Specifies maturity of contract

Underlying assets

Typical underlying assets

- Stocks (Equity)
- Interest rates (Fixed Income products)
- Currencies and commodities

Most common derivative contracts

Forwards/Futures

- Fixes the value of the underlying asset at a future date

Swaps

- An exchange of two cash flows

Options

- A possibility, but not an obligation, to buy/sell an asset at a future date

Structured credit products

- Asset-backed securities (ABSs), collateralized debt obligations (CDOs), mortgage-backed securities (MBSs), collateralized mortgage obligations (CMOs)

Structured notes

- Notes where the payoff depends on the development in some linked asset - a currency, equity, commodity, interest rate..

Contracts

Spot contract - A contract for immediate sale and delivery of an asset.

Forward contract - A contract between two people for the delivery of an asset at a negotiated price on a set date in the future.

Futures contract - A contract similar to a forward contract, except there is an intermediary that creates a standardized contract. The two parties do not have to negotiate the terms of the contract. The intermediary guarantees all trades and provides a secondary market for futures trading.

Futures vs. forwards

Basically the same, however:

Trading:

- Futures are exchange traded
- Forwards are trade Over-The-Counter (OTC)

Content of contract:

- Futures are standardized
- Forwards are custom made to fit the exact need of the client

Payment:

- Futures: The change in the value of the contract is settled daily
- Forwards: Payment takes place at the end of the contract. The contract is typically designed to have zero cost , hence zero-value, at initiation.

Forwards and spot contracts

Basic relationship between forward prices and spot prices for equity securities

$$
\begin{aligned}
F_t &= S_0 \left(1 + r_f - y\right)^t \\
F_t &= \text{forward price on contract of length t years} \\
S_0 &= \text{ today's spot price} \\
r_f &= \text{ risk free rate (annual compounding)} \\
y &= \text{dividend yield (annual)}
\end{aligned}
$$

Equity future formula

Why must the formula hold?

Buy stock index today at price S_0.

Buy forward contract and invest S_0 in a bank account (remember - the forward contract is settled at expiration, i.e. no capital outlay at initiation).

Commodity forward formula

A commodity doesn't pay dividends.
However, there is a "negative income" in the form of storage costs.
There may also be some positive value to holding the physical commodity, in addition to its financial value - convenience yield.

Forward and spot contracts

Basic relationship between forward prices and spot prices for commodities

$$
\begin{aligned}
F_t &= S_0 \left(1 + r_f + sc - cy\right)^t = S_0 \left(1 + r_f - ncy\right)^t \\
F_t &= \text{forward price on contract of length t years} \\
S_0 &= \text{today's spot price} \\
r_f &= \text{risk free rate (annual compounding)} \\
cy &= \text{convenience yield (annual)} \\
sc &= \text{storage cost (annual)} \\
ncy &= cy - sc = \text{net convenience yield}
\end{aligned}
$$

Swaps

A swap is an agreement between two firms to exchange cash flows in the future.
The agreement defines the dates on which cash flows are to be exchanged, and how these cash flows will be calculated.

Interest rate swaps
Currency swaps
Credit default swaps
Equity swaps

Forward contracts are swaps with just one cash flow exchange.

Plain vanilla interest rate swap

Company B agrees to pay company A cash flows equal to interest payments at a predetermined fixed-rate on a notional principal for a given period.
Company A agrees to pay company B cash flows equal to interest payments at a floating-rate on the same notional principal for the same time period.
The principal is not exchanged.

European call and put options

A *European* call option on stock ABC gives its owner the right, but not the obligation, to buy an ABC stock at a prespecified *exercise* or *strike price* on a prespecified excercise date.

A *European* put option on stock ABC gives its owner the right, but not the obligation, to sell an ABC stock at a prespecified *exercise* or *strike price* on a prespecified exercise date.

American call and put options

An *American* call option on stock ABC gives its owner the right, but not the obligation, to buy an ABC stock at a prespecified *exercise* or *strike price* **on or before** a prespecified excercise date.

An *American* put option on stock ABC gives its owner the right, but not the obligation, to sell an ABC stock at a prespecified *exercise* or *strike price* **on or before** a prespecified exercise date.

Terminology

- **Underlying asset** - The asset on which the option is written (a specific stock, commodity, currency).
- **Price of underlying asset** (S_t) - The price of the underlying asset at the time t.
- **Exercise/strike price** (K) - The price at which you buy or sell the asset if exercising the option.
- **Expiration date** (T) - The last date on which the option can be exercised (the only date in the case of European options).
- **Risk-free interest rate** (r_f) - The rate of interest earned on a bank deposit or Treasury bill.
- **Call option price** (c_0) -is the price of a call option.
- **Put option price** (p_0) - is the price of a put option.

Payoff of call at expiration

European call option on stock ABC with exercise price $K = 60$

- Stock price at expiration $S_T = 10 \Rightarrow$ option value $= 0$
- Stock price at expiration $S_T = 30 \Rightarrow$ option value $= 0$
- Stock price at expiration $S_T = 60 \Rightarrow$ option value $= 0$
- Stock price at expiration $S_T = 70 \Rightarrow$ option value $= 10$
- Stock price at expiration $S_T = 90 \Rightarrow$ option value $= 30$

$$\text{Value of call at expiration} = \max\{S_T - K, 0\}$$

Payoff of put at expiration

European put option on stock ABC with exercise price $K = 60$

- Stock price at expiration $S_T = 10 \Rightarrow$ option value $= 50$
- Stock price at expiration $S_T = 30 \Rightarrow$ option value $= 30$
- Stock price at expiration $S_T = 60 \Rightarrow$ option value $= 0$
- Stock price at expiration $S_T = 70 \Rightarrow$ option value $= 0$
- Stock price at expiration $S_T = 90 \Rightarrow$ option value $= 0$

$$\text{Value of put at expiration} = \max\{K - S_T, 0\}$$

Buyer or seller

	Buyer	Seller
Call option	Right to buy asset	Obligation to sell asset
Put option	Right to sell asset	Obligation to buy asset

Moneyness

Let S be the stock price at some point during the life of the option and K be the exercise price.

In the money

- Call option: if $S > K$
- Put option: if $S < K$

Out of the money

- Call option: if $S < K$
- Put option: if $S > K$

At the money

- Call or put option: if $S = K$

Put-call parity

The following holds for non-dividend paying European options:
value of call + present value of exercise price = value of put + share price

$$c_0 + PV(K) = p_0 + S_0$$

buy call + invest PV of exercise price in safe asset = buy put + buy share

value of put = value of call + present value of exercise price −share price

$$p_0 = c_0 + PV(K) - S_0$$

buy put = buy call + invest PV of exercise price in safe asset + sell share

Call option

A call option gives the holder the right to buy one share of a stock for a prespecified price K.

The option can never be worth more than the price of the underlying stock.

$$c_0 \leq S_0 \text{ and } C_0 \leq S_0$$

Put option

A put option gives the holder the right to sell one share of a stock for a prespecified price K.

The option can never be worth more than the exercise price K.

$$P_0 \leq K$$

For European put options, the current value can never be higher than the present value of the exercise price K.

$$p_0 \leq PV\left(K\right)$$

Lower bounds on call options

European option on non-dividend paying stock

- *Portfolio A:* Buy one European call option and place an amount of cash equal to $PV(K)$ in a bank account - $(c_0 + PV(K))$.
- *Portfolio B:* Buy one share of the stock - S_0
    - The value at expiration of portfolio A will always be greater than or equal to the payoff on portfolio B at expiration $\Rightarrow$ portfolio A must be worth at least as much as portfolio B today.

$$\begin{aligned} c_0 + PV(K) &\geq S_0 \\ c_0 &\geq \max(S_0 - PV(K), 0) \end{aligned}$$

American option

It is never optimal to exercise an American call option on a non-dividend-paying stock before the expiration date.

As $C_0 \geq c_0$, if $T > 0$

$$C_0 \geq S_0 - PV(K)$$

Assuming $r_f > 0$

$$C_0 > S_0 - K = \text{value if exercised early}$$

European option

- Portfolio A: buy one European put option and one share - $(p_0 + S_0)$
- Portfolio B: place an amount of cash equal to $PV(K)$ in a bank account - $PV(K)$
  - The value at expiration of portfolio A will always be greater than or equal to the payoff on portfolio B at expiration $\Rightarrow$ portfolio A must be worth at least as much as portfolio B today.

$$
\begin{aligned}
p_0 + S_0 &\geq PV(K) \\
p_0 &\geq \max(PV(K) - S_0, 0)
\end{aligned}
$$

American option

If an American put option is sufficiently deep in the money, it is optimal to exercise early.

$$P_0 \geq K - S_0$$

Put-call relationship for American options

The put-call parity holds only for European options.
For American options the following holds

$$S_0 - K \leq C_0 - P_0 \leq S_0 - PV(K)$$

Lower bound for european calls and puts

- *Portfolio A:* Buy one European call option and place an amount of cash equal to $PV(Div) + PV(K)$ in a bank account
- *Portfolio B:* Buy one share of the stock - S_0
  - The value at expiration of portfolio A will always be greater than or equal to the payoff on portfolio B at expiration $\Rightarrow$ portfolio A must be worth at least as much as portfolio B today.

$$c_0 \geq S_0 - PV(Div) - PV(K)$$

- Portfolio A: buy one European put option and one share - $(p_0 + S_0)$
- Portfolio B: place an amount of cash equal to $PV(Div) + PV(K)$ in a bank account
  - The value at expiration of portfolio A will always be greater than or equal to the payoff on portfolio B at expiration $\Rightarrow$ portfolio A must be worth at least as much as portfolio B today.

$$p_0 \geq PV(Div) + PV(K) - S_0$$

Early exercise of American call options on dividend paying stocks

Under this scenario it might be optimal to exercise American calls prematurely.
The holder of the option is not entitled to any dividend payments during the life og the option.
When a dividend payment is made, the price of the stock drops by an equivalent amount.
Hence it may pay to exercise just before such a dividend payment is made.
However, it is never optimal to exercise between ex-dividend dates

Put-call parity

For European options on dividend paying stocks

$$c_0 + PV\left(Div\right) + PV\left(K\right) = p_0 + S_0$$

For American options on dividend paying stocks

$$S_0 - PV\left(Div\right) - K \leq C_0 - P_0 \leq S_0 - PV\left(K\right)$$

How do we value options?

Standard approach to valuing assets:

- Find expected cash flows
- Discount the cash flows at the opportunity cost of capital

We can in principle find the expected cash flows from an option, but not the opportunity cost of capital.
The risk of the option changes when the price of the underlying asset changes.
How risky the option is, depends on the price of the underlying asset and the exercise price of the option.

Two approaches to pricing options

- Create a package of investments in the underlying stock and a bank account, which exactly replicates the payoffs of the option. The price of this package must equal the price of the option.
- Pretend investors are risk-neutral and use the risk-neutral probability measure to calculate the expected payoff on the option. Discount by the risk-free rate of interest.

No arbitrage

Call option on ABC with exercise price $K = \$60$ and 8 months to expiration.

Current price of ABC stock $S_0 = \$60$.

Risk-free rate $r_f = 1.5\%\ p.a.$

Assume we know that the stock price of ABC at $T = \frac{2}{3}$ can only be either \$45 or \$80.

Payoff from call option:

Stock price at exp.	$S_T = \$45$	$S_T = \$80$
call option payoff	\$0	\$20

Option equivalent

Construct an option equivalent by combining an investment of Δ stocks and investing an amount B at the risk-free interest rate.

Stock price at exp.	$S_T = \$45$	$S_T = \$80$
call option payoff	$\$0$	$\$20$
option equivalent	$\Delta * \$45 + \$B(1+r)^T$	$\Delta\$80 + \$B\,(1+r)^T$
	$= \$0$	$= \$20$

The price of this combination must equal the price of the call option.

$$\begin{aligned} \text{Value of option} &= \Delta * \text{share price} + \text{bank placement} \\ c_0 &= \Delta * S_0 + B \end{aligned}$$

Option equivalent

Payoff's from option equivalent portfolio

$$\begin{aligned} \Delta 80 + B\left(1+0.015\right)^{\frac{2}{3}} &= 20 \\ \Delta 45 + B\left(1+0.015\right)^{\frac{2}{3}} &= 0 \end{aligned}$$

Isolate delta

$$\begin{aligned} \Delta 80 - \Delta 45 &= 20 - 0 \\ \Delta &= \frac{20-0}{(80-45)} = \frac{4}{7} \end{aligned}$$

Compute amount to be placed in bank account

$$\begin{aligned} B\left(1+0.015\right)^{\frac{2}{3}} &= -\frac{4}{7} * 45 \\ B &= -\frac{4}{7} * 45 * \left(1+0.015\right)^{-\frac{2}{3}} = -25.46 \end{aligned}$$

Value of option

$$\begin{aligned}\text{Value of option} &= \Delta * \text{share price} + \text{bank placement} \\ c_0 &= \frac{4}{7} * \$60 - \$25.46 = 8.83\end{aligned}$$

The number of shares needed to replicate one option is called the hedge ratio or option delta:

$$\text{Option delta} = \frac{\text{spread of possible option prices}}{\text{spread of possible share prices}}$$

Risk-neutral valuation

Set up a portfolio by going long Δ stocks and short 1 option, in such a way that there is no uncertainty about the value of the portfolio at expiration. The investment is riskless.

Stock price at exp.	$S_T = \$45$	$S_T = \$80$
call option payoff	\$0	\$20
riskless portfolio	$\Delta * \$45 - \$0 = x$	$\Delta \$80 - \$20 = x$

$$\begin{aligned} \Delta * \$45 - \$0 &= \Delta \$80 - \$20 \\ \Delta &= \frac{\$20 - \$0}{\$80 - \$45} = \frac{4}{7} \end{aligned}$$

Payoff on riskless portfolio

$$\frac{4}{7} * \$45 = \$25.71$$
$$\frac{4}{7} * \$80 - \$20 = \$25.71$$

As this is a riskless portfolio, it must earn the risk-free interest rate. So the price at time 0 must be

$$\$25.71 * (1 + 0.015)^{-\frac{2}{3}} = \$25.46$$

The price of the stock is \$60 at time 0, so the value of the call option is

$$\frac{4}{7} * \$60 - c_0 = \$25.46$$
$$c_0 = \$8.83$$

Generalization

Let the initial stock price be S_0 and the current price of an option on the stock V. The expiration date of the option is T.
During the life of the option, the stock can either move up to a new price of S_0u or down to S_0d $(u > 1; d < 1)$.
If the stock moves up, let the payoff from the option be V_u.
If the stock moves down, let the payoff from the option be V_d.
The value of the portfolio (long Δ stocks and short 1 option) at expiration will be

$$S_0 u\Delta - V_u$$

if the stock moves up. And

$$S_0 d\Delta - V_d$$

if the stock moves down.

These will be equal when

$$\Delta = \frac{V_u - V_d}{S_0 u - S_0 d}$$

When delta takes on this value, the portfolio is riskless and must earn the risk-free interest rate. At time 0 the value of the portfolio is therefore

$$\left(S_0 u \Delta - V_u\right)\left(1 + r_f\right)^{-T}$$

The initial cost of the portfolio is

$$S_0 \Delta - V$$

Hence

$$
\begin{aligned}
\left(S_0 u \Delta - V_u\right)\left(1+r_f\right)^{-T} &= S_0 \Delta - V \\
V &= S_0 \Delta - \left(S_0 u \Delta - V_u\right)\left(1+r_f\right)^{-T} \\
&= S_0 \frac{V_u - V_d}{S_0 u - S_0 d} \\
&\quad - \left(S_0 u \frac{V_u - V_d}{S_0 u - S_0 d} - V_u\right)\left(1+r_f\right)^{-T} \\
&= \left(1+r_f\right)^{-T}\left[q V_u + (1-q) V_d\right] \\
q &= \frac{\left(1+r_f\right)^{T} - d}{u - d}
\end{aligned}
$$

we can interpret q as the probability of an up movement in the stock and $1 - q$ as the probability of a down movement.
The value of the option today is then the expected future value discounted at the risk-free rate.

The expected return on the stock, when q is assumed to be the probability of an up movement

$$\begin{aligned} E\left(S_T\right) &= qS_0u + (1-q)\,S_0d \\ &= qS_0(u-d) + S_0d \\ &= \frac{\left(1+r_f\right)^T - d}{u-d} S_0(u-d) + S_0d \\ &= S_0\left(1+r_f\right)^T \end{aligned}$$

Setting the probability of an up movement in the stock to q is thus equivalent to the rate of return on the stock being the risk-free rate.
This thus assumes that individuals are indifferent to risk - a risk-neutral world.
Using the above q measure is an example of risk-neutral valuation.