Fixed Income Basics II

Corporate Finance and Incentives

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Practical stuff

- I made a mistake!
- Correct answer is: $(1 + r_{0,2})^2 = (1 + r_{0,1}) * (1 + r_{1,2})$
- ...and the winner is...
- Remember to sign up for exam. Last chance this week!

Spot rates

The spot rate is the rate of return on a spot loan a.k.a. a zero coupon bond.

Spot rates are also known as the yield to maturity on zero coupon bonds

$$y\left(0, t\right) = \left(\frac{1}{d_t}\right)^{\frac{1}{t}} - 1$$

Here we are assuming that t is measured in years and that the spot rates are annually compounded rates.

The relationship between spot rates and maturity, or term, is known as the term structure of interest rates

Term structure

The most common shape of the yield curve is upward sloping

- higher interest rates at longer maturities than at short maturities.

Sometimes the yield curve is downward sloping (inverted)

- lower interest rates at longer maturities than at short maturities.

And sometimes it's just flat

- same interest rate at all maturities.

What determines the term structure

One way to investigate, what the current term structure implies is to look at forward rates.

A forward interest rate is the interest rate implied by the current spot rates for a specified future time period.
The one-year forward rate in one year would be the interest rate that makes an investment in a current two-year zero-coupon bond equal to an investment today in a 1 year zero-coupon bond followed by an investment in a 1 year zero-coupon bond one year from now.

$$
\begin{aligned}
(1+r_2)^2 &= (1+r_1) * (1+f(0,2)) \\
f(0,2) &= \frac{(1+r_2)^2}{(1+r_1)} - 1
\end{aligned}
$$

Forward rates

$$
\begin{aligned}
(1+r_3)^3 &= (1+r_2)^2 * (1+f(0,3)) \\
f(0,3) &= \frac{(1+r_3)^3}{(1+r_2)^2} - 1 \\
(1+r_4)^4 &= (1+r_3)^3 * (1+f(0,4)) \\
f(0,4) &= \frac{(1+r_4)^4}{(1+r_3)^3} - 1
\end{aligned}
$$

In terms of discount factors

$$
f(0,t) = \frac{d_{t-1}}{d_t} - 1
$$

Yield curves

- To give a quick idea of what the term structure looks like, yield curves are still usefull.
- Yield curve - curve of yields on government bonds with varying maturities.
- We have seen that bonds with the same time to maturity can have different yields - depending on the size of coupon payments.
- We need a better measure of "maturity".

Pricing

- Once we have the discount factors, we can price any cashflow
- ZCBs are only traded for very short maturities – although long maturity ZCBs have been created synthetically by separating (termed stripping) the parts of other bonds and selling them separately – at least in the USA
- 3 most common real bonds
  1. Annuities ("annuiteter") – payment is the same in all periods – e.g. home mortgages
  2. Serial bonds ("serielån") – reduction in principal is the same in all periods – quite rare & somewhat antiquated
  3. Bullet bonds ("stående lån") – only interest is paid before maturity – by far the most common type; issued by companies and governments

Macaulay duration

If a bond makes coupon payments, not all payments on the bond will occur at maturity.
The average time to each cash flow will be less than the time to maturity.
To measure this "average time to maturity" we can use the Macaulay duration measure

$$
\begin{aligned}
D_{Mac} &= \frac{1}{PV(C,y)} \sum_{t=1}^{T} t \frac{c_t}{(1+y)^t} \\
&= \frac{1 * PV(c_1, y)}{PV(C,y)} + \frac{2 * PV(c_2, y)}{PV(C,y)} + \frac{3 * PV(c_3, y)}{PV(C,y)} + ..
\end{aligned}
$$

where $PV(c,y)$ is the value or price of the bond i.e. the sum of the present value of all payments on the bond

$$\begin{aligned} PV(c,y) &= \frac{c_1}{(1+y)} + \frac{c_2}{(1+y)^2} + \frac{c_3}{(1+y)^3} + \dots \\ &= \frac{c_1}{(1+r_1)} + \frac{c_2}{(1+r_2)^2} + \frac{c_3}{(1+r_3)^3} + \dots \end{aligned}$$

And

$$PV(c_x, y) = \frac{c_x}{(1+y)^x}$$

Duration is thus a weighted average of the times at which payments are recieved (1, 2, 3,..). The weight is the proportion of the bond's total present value stemming from the payment at each of these times.
The weights will sum to one.
Note, this duration measure is based on the yield to maturity of the bond or alternatively a flat term structure.

How are bond prices affected by changes in the level of interest rates

- When we talk about "the interest rate in the economy" we refer to the short-term rate set by central banks.
- When we talk about "interest rates in the economy" we refer to the spot rates shown previously.
- If we talk about the interest rate of a bond we refer to the yield on the bond.
- We could be interested to know how bond prices, of varying duration, are affected by changes in the level of interest rates.
  - f.x. a 1% increase in the level of interest rates refers to a 1% increase in spot rates at all maturities.
  - this is equivalent to a 1% increase in the yield on the bond.

Modified duration

How does a 1% change in the level of interest rates affect the price of bonds?
We can use the bonds modified duration or volatility

$$\text{Modified duration} = \frac{D_{Mac}}{(1+y)}$$

How is the bond price affected by a change in the yield to maturity (level change in spot rates)?

$$\begin{aligned}
\frac{\partial PV(C,y)}{\partial y} &= \sum_{t=1}^{T} -t \frac{c_t}{(1+y)^{t+1}} \\
&= -\frac{D_{Mac} * PV(C,y)}{(1+y)} \\
&= -D_{Mod} * PV(C,y)
\end{aligned}$$

Modified duration

If we make a small parallel shift to the yield curve, and increase all interest rates by Δy

$$\begin{aligned}
\frac{\Delta PV(C,y)}{\Delta y} &= -D_{Mod} * PV(C,y) \\
\frac{\Delta PV(C,y)}{PV(C,y)} &= -D_{Mod} * \Delta y
\end{aligned}$$

- The modified duration thus gives an indication of how much the bond price will be affected by a change in interest rates.
- The bonds modified duration changes as the interest rate changes.
- Modified duration is higher at low interest rates than at high interest rates.

Convexity

Convexity measures how interest rate sensitivity changes with interest rates

$$k(C, y) = \frac{1}{PV(C, y)} \sum_{t=1}^{T} t^2 \frac{c_t}{(1+y)^t}$$

It is the curvature of the price-yield relation - the second derivative of the price function

Compounding

- So far, our model has been very stylized: dates $1, 2, \ldots, T$
- Would often be interpreted as years
- But: often we face dates that are not integer multiples of the fundamental time unit: a bond can have payments twice a year, and a loan can have daily compounding
- The fact that a rate r_m is compounded m times per time unit (year) means that every $1/m$ years, r_m/m is added to the deposit/debt, and from that point on, further interest is calculated based on the deposit including that amount
- More frequent compounding thus means a higher effective interest rate, as interest is then calculated based on a larger amount

Compounding

Assume an amount A invested for n years at an interest rate of r per annum.
If the interest rate is compounded once per year, the terminal value of the investment is

$$A(1+r)^n$$

If the interest rate is compounded m times per year, the terminal value of the investment is

$$A\left(1+\frac{r}{m}\right)^{mn}$$

In the limit, as $m \to \infty$, i.e. continuous compounding, the terminal vaue is

$$Ae^{rn}$$

Compounding

Let r_c be a rate of interest with continous compounding and r_m be the equivalent rate with a compounding m times per annum.

$$e^{r_c} = \left(1 + \frac{r_m}{m}\right)^m$$

Hence

$$\begin{aligned} r_c &= m \ln\left(1 + \frac{r_m}{m}\right) \\ r_m &= m\left(e^{r_m/m} - 1\right) \end{aligned}$$

Accrued interest

- Assume that you buy a bond today, that pays coupon every year on November 11
- You would then have to compensate the seller for the interest that he/she has earned from November 11 last year until today
- This means that bond prices actually jump up and down systematically: jump down whenever a coupon was paid
- We don't like the look of that

Accrued interest

- To avoid prices jumping up and down, we use the concept accrued interest
- In stead of the price of the bond reflecting the interest accrued between two coupon dates, the buyer of the bond simply pays the seller the accrued interest on top of the price
- This means that the price is calculated net the accrued interest; thus it does not jump up and down

Accrued interest

$$\text{Accrued interest} = \frac{\#\text{days since last coupon}}{\#\text{days between coupons}} \text{Coupon}$$

We use the terms:

- Clean price: Price not including accrued interest (i.e. the one you would read on the financial pages)
- Dirty price: Clean price + accrued interest (what you actually pay)